

# A PRIMER TO SLOW LIGHT

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## Abstract

Laboratory-based optical analogs of astronomical objects such as black holes rely on the creation of light with an extremely low or even vanishing group velocity (slow light). These brief notes represent a pedagogical attempt towards elucidating this extraordinary form of light. This paper is a contribution to the book *Artificial Black Holes* edited by Mario Novello, Matt Visser and Grigori Volovik. The paper is intended as a primer, an introduction to the subject for non-experts, not as a detailed literature review.

# 1 Motivation

Creating a black hole with humble human resources is certainly a fantastic idea, yet perhaps not entirely lunatic. Recent experimental progress in quantum optics, in particular the generation of slow light [1, 2, 3] has opened a route towards the formation of optical event horizons for light or towards other less dramatic but equally interesting phenomena. Most of the ideas have been discussed in a series of papers [4] and review articles [5]. The underlying atomic and optical physics is perhaps less familiar to the audience of this book. Therefore, it might be worthwhile to develop some of the physics behind slow light from basic principles that are hopefully known to most readers.

## 2 Light-matter interaction

Consider an atomic medium capable of interacting with light, for example a cell filled with Rubidium vapor or one of the alkali Bose–Einstein condensates. The atoms that constitute the medium are electrically neutral but polarizable by light. Since atoms are usually much smaller than an optical wavelength, light experiences the atoms as dipoles (as the lowest order in the multipole expansion of an electrically neutral charge distribution [6]). In the dipole approximation, the energy density of the light-matter interaction is given by the negative scalar product  $-\mathbf{P} \cdot \mathbf{E}$  of the matter polarization  $\mathbf{P}$  (dipole-moment density) and the electric field  $\mathbf{E}$  [6]. Therefore, the Lagrangian of light and atomic matter reads in SI units

$$\mathcal{L} = \frac{\varepsilon_0}{2} (E^2 - c^2 B^2) + \mathbf{P} \cdot \mathbf{E} + \mathcal{L}_A. \quad (1)$$

The first term describes the free electromagnetic field with  $c$  denoting the speed of light in vacuum. The last term of the Lagrangian characterizes the internal dynamics of the atoms. We represent the electric field  $\mathbf{E}$  and the magnetic field  $\mathbf{B}$  in terms of the vector potential  $\mathbf{A}$  in Coulomb gauge [6]

$$\mathbf{E} = -\partial_t \mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \quad \nabla \cdot \mathbf{A} = 0. \quad (2)$$

Let us assume, for simplicity, that both the electromagnetic field and the medium are uniform in two spatial directions in Cartesian coordinates, but may vary in the third direction  $z$ . Furthermore, we consider light with fixed polarization so that we can concentrate on one component  $A$  of the vector potential  $\mathbf{A}$ . The Lagrangian simplifies to

$$\mathcal{L} = \frac{\varepsilon_0}{2} ((\partial_t A)^2 - c^2 (\partial_z A)^2) - P \partial_t A + \mathcal{L}_A. \quad (3)$$

We obtain from the Euler–Lagrange equation the wave equation

$$(\partial_t^2 - c^2 \partial_z^2) E = -\frac{1}{\varepsilon_0} \partial_t^2 P. \quad (4)$$

Atoms have a well-defined level structure such that light, oscillating  $10^{15}$  times per second, must match the atomic transition frequencies, because otherwise the effect

of  $\mathbf{E}$  in  $-\mathbf{P} \cdot \mathbf{E}$  is rapidly washed out. Given a certain frequency range of light, the optical field thus interacts with a selected few of the atomic levels, which greatly simplifies matters.

Let us describe the atoms quantum-mechanically, while regarding the electromagnetic field as classical. The simplest relevant atomic system involves just two levels, say the ground state  $|a\rangle$  and the excited state  $|b\rangle$ . The Hamiltonian of the two-level atom is simply

$$\hat{H} = \hbar\omega_{ab} |b\rangle\langle b| - \frac{\kappa_{ab}}{2} (\hat{A} + \hat{A}^\dagger) E, \quad (5)$$

$$\hat{A} = |a\rangle\langle b|, \quad (6)$$

where  $\omega_{ab}$  denotes the atomic transition frequency and  $\kappa_{ab}$  corresponds to the atomic dipole moment, a real number for transitions between bound states. In the Heisenberg picture the transition operator  $\hat{A}$  oscillates with positive frequencies near  $\omega_{ab}$ . Therefore,  $\hat{A}$  couples entirely to the negative-frequency component  $E^{(-)}$  of the electric field, while  $\hat{A}^\dagger$  couples to the positive-frequency component  $E^{(+)}$ . We arrive at the effective Hamiltonian

$$\hat{H} = \hbar\omega_{ab} |b\rangle\langle b| - \frac{\kappa_{ab}}{2} (\hat{A} E^{(-)} + \hat{A}^\dagger E^{(+)}) . \quad (7)$$

To describe the quantum state of the atom, we employ a density matrix  $\hat{\rho}$  that characterizes a statistical ensemble of pure states  $|\psi_a\rangle$  with probabilities  $p_a$ ,

$$\hat{\rho} = \sum_a p_a |\psi_a\rangle\langle\psi_a|. \quad (8)$$

Probabilities are non-negative and sum up to unity. Consequently, the density matrix has non-negative eigenvalues and is normalized as  $\text{tr}\hat{\rho} = 1$ . In the Schrödinger picture, the density matrix evolves while the operators are invariant in time. According to Lindblad's theorem [7], the evolution of a normalized and non-negative density matrix is governed by the master equation [8]

$$\frac{d\hat{\rho}}{dt} = \frac{i}{\hbar} [\hat{\rho}, \hat{H}] - \sum_l \gamma_l (\hat{A}_l^\dagger \hat{A}_l \hat{\rho} - 2\hat{A}_l \hat{\rho} \hat{A}_l^\dagger + \hat{\rho} \hat{A}_l^\dagger \hat{A}_l) . \quad (9)$$

The Lindblad operators  $\hat{A}_l$  describe dissipative processes occurring at the rates  $\gamma_l$ , for example the spontaneous emission from the excited state to the ground state. As a result of the light-matter interaction, a medium of  $n_A$  atoms per volume generates a matter polarization of

$$P = n_A \frac{\kappa_{ab}}{2} \text{tr} \{ \hat{\rho} (\hat{A} + \hat{A}^\dagger) \} = n_A \kappa_{ab} \text{Re} \langle a | \hat{\rho} | b \rangle . \quad (10)$$

In this way the response of the atoms to the electric field modifies the propagation of light given by the wave equation (4).

Consider a medium at rest in a regime of linear response. Here the medium integrates the local history of the electric field via the susceptibility  $\chi$ ,

$$P = \varepsilon_0 \int_{-\infty}^{+\infty} \chi(t-t') E(t') dt' . \quad (11)$$

Causality implies that  $P$  must not depend on the future of the  $E$  field, which restricts the integral (11) to the time interval  $(-\infty, 0]$  by requiring

$$\chi(t) = 0 \quad \text{for} \quad t < 0. \quad (12)$$

Consider the Fourier-transformed (spectral) susceptibility

$$\tilde{\chi}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \chi(t) e^{i\omega t} dt = \frac{1}{2\pi} \int_0^{+\infty} \chi(t) e^{i\omega t} dt. \quad (13)$$

Because  $\chi(t)$  is real, we get

$$\tilde{\chi}(-\omega) = \tilde{\chi}^*(\omega). \quad (14)$$

Let us regard  $\tilde{\chi}(\omega)$  as a function of complex frequency  $\omega$ . When the imaginary part of  $\omega$  is positive,  $\tilde{\chi}(\omega)$  cannot have singularities, because here the Fourier integral contains a factor  $\exp(-[\text{Im } \omega]t)$  that enforces convergence. Therefore,  $\tilde{\chi}(\omega)$  is analytic on the upper half plane. Causality thus implies analyticity [9]. For a non-dispersive medium  $\tilde{\chi}(\omega)$  is constant over the relevant frequency range and  $\chi(t)$  is reduced to a delta function, describing an instant response of the medium. In dispersive media,  $\tilde{\chi}(\omega)$  varies and the poles of  $\tilde{\chi}(\omega)$  on the lower half plane correspond to atomic resonances. At the real  $\omega$  axis the real part of  $\tilde{\chi}$  describes the dispersive properties of the medium, whereas the imaginary part accounts for dissipation. Given an analytic function  $\tilde{\chi}(\omega)$  on the upper half plane that decays sufficiently fast when  $\omega \rightarrow \infty$ , the real and imaginary parts of  $\tilde{\chi}(\omega)$  at the real  $\omega$  axis are related to each other by Hilbert transformations [10] (Kramers-Kronig relations [9]). The imaginary part of  $\tilde{\chi}(\omega)$  is thus uniquely determined by the real part and vice versa.

Dispersion influences the group velocity of light pulses. Suppose that the medium properties do not vary significantly within the scale of an optical wavelength. In this case we can characterize completely the propagation of light pulses by the dispersion relation between the wave number  $k$  and the frequency  $\omega$ ,

$$k^2 = \frac{\omega^2}{c^2} [1 + \tilde{\chi}(\omega, z)], \quad (15)$$

derived from the wave equation by Fourier transformation with respect to space and time. A light pulse propagates like a particle with Hamiltonian  $\omega$  and momentum  $k$ , subject to Hamilton's equations

$$\frac{dk}{dt} = -\frac{\partial \omega}{\partial z}, \quad \frac{dz}{dt} = \frac{\partial \omega}{\partial k}. \quad (16)$$

The group velocity  $v_g$  is the velocity  $dz/dt$  of the fictitious light particle,

$$v_g = \frac{\partial \omega}{\partial k} = \left( \frac{\partial k}{\partial \omega} \right)^{-1} = \frac{c}{n + \frac{\omega}{2n} \frac{\partial \tilde{\chi}}{\partial \omega}}, \quad n = \sqrt{1 + \tilde{\chi}}. \quad (17)$$

Slow light [1] involves an extremely dispersive medium where  $\partial \tilde{\chi} / \partial \omega$  is large and, consequently, where the group velocity is very low (a few meters per second, or lower).

### 3 Ordinary media

Before we embark on discussing extremely dispersive media, let us consider an ordinary medium composed of two-level atoms at rest. Assume that the atoms are identical, with equal transition frequencies  $\omega_{ab}$ , and that they are affected by dissipative relaxation processes. The dissipation transfers excitations from the excited to the ground states, described by the Lindblad operator (6). Assume that the transition rate  $\gamma$  dominates the time scale of the light-atom interaction. In this regime, relaxation forces the atomic dipoles to follow the fields. In the case of linear response we can decompose the electric field into Fourier components. To analyse the response, it is sufficient to study the reaction of one of the atoms to a single monochromatic wave with frequency  $\omega$ . We characterize the field-strength of the wave in terms of the Rabi frequency  $\Omega$  with

$$\Omega e^{-i\omega t} = \frac{\kappa_{ab}}{\hbar} E^{(+)} . \quad (18)$$

In the absence of relaxation, an atom would oscillate between the ground and the excited state with frequency  $\Omega$  (Rabi flopping). On the other hand, relaxation leads to a stationary state. To describe the stationary regime we use an appropriate interaction picture.

In an interaction picture, indicated by tildes over operators, the dynamics with respect to a partial Hamiltonian  $\hat{H}_0$  is separated from the total evolution of the density matrix,

$$\tilde{\rho} = \hat{U}_0^\dagger \hat{\rho} \hat{U}_0, \quad \hat{U}_0 = \exp\left(-\frac{i}{\hbar} \hat{H}_0 t\right) . \quad (19)$$

To derive the evolution equation, we differentiate  $\tilde{\rho}$  with respect to  $t$  and apply the master equation (9). Suppose that the commutator between  $\hat{A}_l$  and  $\hat{H}_0$  is proportional to  $\hat{A}_l$ ,

$$[\hat{A}_l, \hat{H}_0] = \hbar\omega_l \hat{A}_l . \quad (20)$$

In this case  $\hat{U}_0^\dagger \hat{A}_l \hat{U}_0$  gives  $\hat{A}_l \exp(-i\omega_l t)$ , and thus the dissipative part of the master equation (9) remains the same in the interaction picture. The Hamiltonian is transformed according to

$$\tilde{H} = \hat{U}_0^\dagger \hat{H} \hat{U}_0 - \hat{H}_0 , \quad (21)$$

such that we obtain

$$\frac{d\tilde{\rho}}{dt} = \frac{i}{\hbar} [\tilde{\rho}, \tilde{H}] - \sum_l \gamma_l \left( \hat{A}_l^\dagger \hat{A}_l \tilde{\rho} - 2\hat{A}_l \tilde{\rho} \hat{A}_l^\dagger + \tilde{\rho} \hat{A}_l^\dagger \hat{A}_l \right) . \quad (22)$$

Returning to the two-level atom, we use the interaction picture with respect to

$$\hat{H}_0 = \hbar\omega |b\rangle \langle b| \quad (23)$$

that preserves the dissipative dynamics and leads to the time-independent Hamiltonian

$$\tilde{H} = \hbar(\omega_{ab} - \omega) |b\rangle \langle b| - \frac{1}{2} \left( \hat{A} \Omega^* + \hat{A}^\dagger \Omega \right) . \quad (24)$$

The atom reaches a stationary state when the relaxation balances the optical transition,

$$\frac{i}{\hbar} [\tilde{\rho}, \tilde{H}] = \gamma \left( \hat{A}^\dagger \hat{A} \tilde{\rho} - 2 \hat{A} \tilde{\rho} \hat{A}^\dagger + \tilde{\rho} \hat{A}^\dagger \hat{A} \right). \quad (25)$$

Given the normalization  $\text{tr} \hat{\rho} = 1$ , we can easily solve the linear equation (25) for the density-matrix components. Assuming linear response, we linearize the solution  $\tilde{\rho}$  in the Rabi frequency  $\Omega$  and obtain

$$\tilde{\rho} = \begin{bmatrix} 1 & -\frac{\Omega^*}{2(\omega - \omega_{ab} - i\gamma)} \\ -\frac{\Omega}{2(\omega - \omega_{ab} + i\gamma)} & 0 \end{bmatrix}. \quad (26)$$

The positive-frequency component of the matter polarization (10) is therefore

$$P^{(+)} = \frac{n_A}{2} \kappa_{ab} \langle b | \hat{U}_0 \tilde{\rho} \hat{U}_0^\dagger | a \rangle = \frac{n_A}{4} \kappa_{ab} \tau_{ab} \Omega e^{-i\omega t} \quad (27)$$

where we have introduced

$$\tau_{ab} = -\frac{1}{\omega - \omega_{ab} + i\gamma}, \quad \tau_{ba} = -\frac{1}{\omega + \omega_{ab} + i\gamma}. \quad (28)$$

For monochromatic light,  $P^{(+)}$  gives simply  $\varepsilon_0 \tilde{\chi} E^{(+)}$ . Considering the property (14) of the spectral susceptibility we obtain the Lorentzian

$$\tilde{\chi} = \frac{n_A}{4} \frac{\kappa_{ab}^2}{\varepsilon_0 \hbar} (\tau_{ab} + \tau_{ba}). \quad (29)$$

Transforming to a real-time susceptibility  $\chi(t)$  we see easily that  $\chi(t)$  responds within the relaxation time  $\gamma$ . In accordance with causality, the Fourier-transformed susceptibility  $\tilde{\chi}$  is analytic on the upper half plane. The single poles at  $\pm\omega_{ab} - i\gamma$  correspond to the atomic two-level resonance. On the real frequency axis,  $\tilde{\chi}$  is peaked at  $\pm\omega_{ab}$  with the spectral line width  $\gamma$ . Figure 1 illustrates how the real and the imaginary part of the spectral susceptibility depend on the frequency. The medium is most dispersive near the resonance frequency  $\omega_{ab}$  where, unfortunately, it is also most absorptive. Instead of slowing down light, the medium turns completely opaque.

## 4 Electromagnetically-Induced Transparency

Electromagnetically-Induced Transparency (EIT) [11] has served as a method to slow down light significantly [1] or, ultimately, to freeze light completely [2, 3]. Like other successful techniques, EIT is based on a simple idea [11]: A control beam of laser light couples the upper levels of an atom, and, in this way, the beam strongly modifies the optical properties of the atom. In particular, the coupling of the excited states affects the transition from the atomic ground state to one of the upper states, *i.e.*, the ability of the atom to absorb probe photons with matching transition frequency. Destructive quantum interference between the paths of the transition

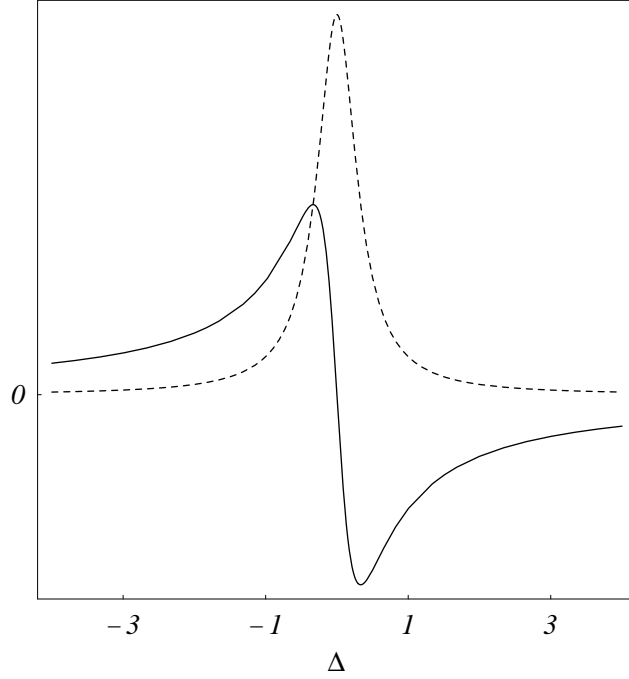


Figure 1: Spectral susceptibility of light in an ordinary dielectric medium. The figure shows the real part (solid line) and the imaginary part (dashed line) of  $\tilde{\chi}(\omega)$  as a function of the detuning  $\Delta = \omega - \omega_0$  in arbitrary units. The function is given by equations (28) and (29). The line width  $\gamma$  was set to  $1/3$  in the units used. The imaginary part of  $\tilde{\chi}(\omega)$  has a peak at the atomic resonance frequency (for zero detuning). Outside the resonance the real part grows monotonically, corresponding, according to equation (17), to a positive group velocity. Near the resonance the dispersion reaches a maximum. At the resonance the real part of  $\tilde{\chi}(\omega)$  decreases sharply (anomalous dispersion leading to a negative group velocity). However, the interesting spectral region of low or negative group velocity is totally overshadowed by absorption.

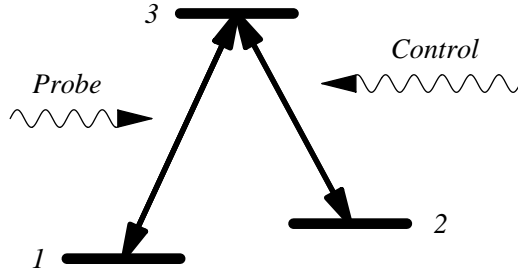


Figure 2: *Three-level atom in a regime of Electromagnetically-Induced Transparency. The control beam couples the levels 2 and 3, which influences strongly the optical properties of the atom for a weaker probe beam tuned to the transition  $1 \leftrightarrow 3$ .*

process turns out to eliminate absorption at exact resonance [11]. A medium composed of such optically-manipulated atoms is transparent in a spectral region where it would otherwise be completely opaque. In the vicinity of the transparency frequency  $\omega_0$  the medium is highly dispersive, *i.e.*, the refractive index changes within a narrow frequency interval. In turn, probe light pulses with a carrier frequency at  $\omega_0$  travel with a very low group velocity  $v_g$  [12].

Consider the three-level atom illustrated in figure 2. The atom is characterized by the energy-level differences  $\hbar\omega_{12}$  and  $\hbar\omega_{23}$  with  $\omega_{12} + \omega_{23} = \omega_{13} \equiv \omega_0$ . Typically, the transition frequencies  $\omega_{13}$  and  $\omega_{23}$  are in the optical range of the spectrum or in the near infrared ( $10^{15}\text{Hz}$ ), whereas the frequency  $\omega_{12}$  is much lower ( $10^9\text{Hz}$ ). The atom is subject to fast relaxation mechanisms ( $10^6\text{Hz}$ ) that transport atomic excitations from the  $|3\rangle$  state down to  $|1\rangle$  and from  $|3\rangle$  to  $|2\rangle$ , mainly caused by spontaneous emission. Hardly any excitations move from  $|2\rangle$  to  $|1\rangle$ , because the spontaneous emission rate is proportional to the cube of the frequency [13]. Here the relaxation may be dominated by other processes, for instance by spin-exchanging collisions. Without relaxation the dynamics of the atom is governed by the Hamiltonian

$$\hat{H} = \begin{bmatrix} 0 & 0 & -\frac{1}{2} \kappa_{13} E_p^{(-)} \\ 0 & \hbar\omega_{12} & -\frac{1}{2} \kappa_{23} E_c^{(-)} \\ -\frac{1}{2} \kappa_{13} E_p^{(+)} & -\frac{1}{2} \kappa_{23} E_c^{(+)} & \hbar\omega_{13} \end{bmatrix}. \quad (30)$$

The Hamiltonian represents the atomic level structure and describes the  $-\hat{P}E$  interaction with light, considering here only the frequency components  $E_p^{(\pm)}$  and  $E_c^{(\pm)}$  that match approximately the level structure. The  $E_p$  and  $E_c$  fields are the probe and control light respectively. We describe relaxation phenomenologically by the flip processes

$$\hat{A}_1 = |1\rangle\langle 3|, \quad \hat{A}_2 = |2\rangle\langle 3|, \quad (31)$$



occurring at the rates  $\gamma_1$  and  $\gamma_2$ , typically a few  $10^6$  Hz. Suppose that the three-level atom is illuminated with monochromatic control light at frequency  $\omega_c$  in exact resonance with the  $2 \leftrightarrow 3$  transition,

$$\omega_c = \omega_{23} . \quad (32)$$

Consider a regime of linear response. In this case we can decompose the probe field into monochromatic waves, to describe completely the reaction of the atom. We characterized the two light fields involved by their Rabi frequencies  $\Omega_c$  and  $\Omega_p$ , defined as

$$\Omega_c e^{-i\omega_c t} = \frac{\kappa_{23}}{\hbar} E_c^{(+)} , \quad \Omega_p e^{-i\omega t} = \frac{\kappa_{13}}{\hbar} E_p^{(+)} . \quad (33)$$

The Rabi frequencies set the time scales of atomic transitions caused by the applied light fields. The control beam shall dominate all processes,

$$|\Omega_c| \gg |\Omega_p| , \gamma_1 , \gamma_2 . \quad (34)$$

Mediated by relaxation, the atomic dipoles lose any initial oscillations they might have possessed and follow the optical fields. To find the stationary state, we utilize an interaction picture generated by

$$\hat{H}_0 = \hbar \begin{bmatrix} 0 & 0 & 0 \\ 0 & \omega - \omega_c & 0 \\ 0 & 0 & \omega \end{bmatrix} . \quad (35)$$

Due to commutation relations of the type (20) the dissipative part of the master equation is not changed in the interaction picture. The transformed Hamiltonian has become time-independent,

$$\tilde{H} = -\hbar \begin{bmatrix} 0 & 0 & \frac{1}{2} \Omega_p^* \\ 0 & \omega - \omega_0 & \frac{1}{2} \Omega_c^* \\ \frac{1}{2} \Omega_p & \frac{1}{2} \Omega_c & \omega - \omega_0 \end{bmatrix} . \quad (36)$$

Similar to a two-level atom in a stationary state (25), the optical transitions should balance the relaxation processes,

$$\frac{i}{\hbar} [\tilde{\rho}, \tilde{H}] = \sum_{l=1}^2 \gamma_l \left( \hat{A}_l^\dagger \hat{A}_l \tilde{\rho} - 2 \hat{A}_l \tilde{\rho} \hat{A}_l^\dagger + \tilde{\rho} \hat{A}_l^\dagger \hat{A}_l \right) . \quad (37)$$

We could solve exactly the linear equation (37) for the matrix elements of  $\tilde{\rho}$  with  $\text{tr} \tilde{\rho} = 1$  (using computerized formula manipulation, for example), but without gaining much insight. Fortunately, since we are interested in the regime (34), we can find transparent approximations. Suppose first that also

$$|\Omega_c| \gg |\omega - \omega_0| . \quad (38)$$

We expand the solution of equation (37) to quadratic order in the small quantities (34) and (38), and get

$$\tilde{\rho} = \begin{bmatrix} 1 - \frac{|\Omega_p|^2}{|\Omega_c|^2} & -\frac{\Omega_p^*}{\Omega_c^*} & \frac{2(\omega - \omega_0)}{|\Omega_c|^2} \Omega_p^* \\ -\frac{\Omega_p}{\Omega_c} & \frac{|\Omega_p|^2}{|\Omega_c|^2} & 0 \\ \frac{2(\omega - \omega_0)}{|\Omega_c|^2} \Omega_p & 0 & 0 \end{bmatrix}. \quad (39)$$

We proceed similarly to our analysis of the two-level atom and find, in the positive-frequency range, the spectral susceptibility

$$\tilde{\chi} = \frac{2\alpha}{\omega_0} (\omega - \omega_0), \quad (40)$$

given here in terms of the parameter

$$\alpha = \frac{n_A}{2} \frac{\kappa_{13}^2}{\kappa_{12}^2} \frac{\hbar\omega_0}{\varepsilon_0 |E_c|^2}. \quad (41)$$

The spectral susceptibility  $\tilde{\chi}$  depends linearly on the detuning  $\omega - \omega_0$  and vanishes at the resonance frequency. Here the phase velocity of light is exactly the speed of light in vacuum,  $c$ , but the group velocity (17) is reduced by  $(1 + \alpha)$ ,

$$v_g = \frac{c}{1 + \alpha}. \quad (42)$$

We call the  $\alpha$  parameter (41) group index. The parameter is proportional to the ratio between the probe-light energy per photon,  $\hbar\omega_0$ , and the control-light energy per atom,  $\varepsilon_0 |E_c|^2/n_A$ . Consequently, the less intense the control beam is the slower the probe light is, a paradoxical behavior. Taken to the extreme, the group velocity would vanish when the control beam is totally dimmed. However, in the stationary regime that we are considering, the control beam should dominate (34) and the detuning should be small compared with the modulus of the control's Rabi frequency.

Apparently, the linear spectral slope (40) of the susceptibility ought to be limited. To find the limitation, we expand the exact stationary state of the master equation (22) to linear order in  $\Omega_p$ , in accordance with a regime of linear response. We obtain a spectral susceptibility (29) with

$$\tau_{13} = -\frac{\omega - \omega_0}{(\omega - \omega_0)^2 + i(\omega - \omega_0)(\gamma_1 + \gamma_2) - \frac{1}{4}|\Omega_c|^2}. \quad (43)$$

One can easily verify that the poles of  $\tilde{\chi}$  are located on the lower half plane of the complex frequency  $\omega$ , in agreement with the causality of the medium's response. We expand formula (43) in powers of  $(\omega - \omega_0)/|\Omega_c|$  and see that the medium becomes dissipative when the condition

$$|\omega - \omega_0| \ll \frac{|\Omega_c|^2}{\gamma_1 + \gamma_2} \quad (44)$$

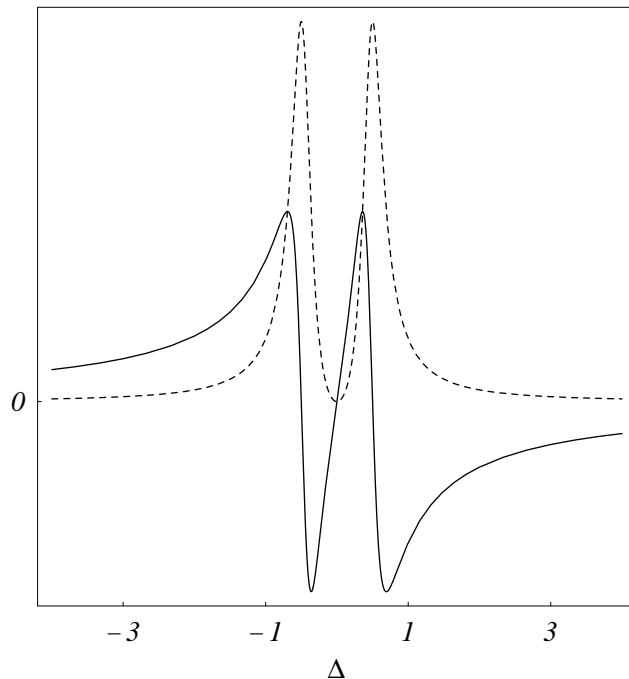


Figure 3: *Susceptibility of the probe light in a medium with Electromagnetically-Induced Transparency.* The figure shows the real part (solid line) and the imaginary part (dashed line) of the spectral susceptibility  $\tilde{\chi}(\omega)$  as a function of the detuning  $\Delta = \omega - \omega_0$  in units of the Rabi frequency  $\Omega_c$  of the control beam, given by equations (29) and (43). The line width  $\gamma_1 + \gamma_2$  was set to  $\Omega_c/3$ . The parameters used agree with the ones in figure 1. Comparing the two figures, we see that EIT radically alters the susceptibility in a spectral region around the probe resonance  $\omega_0$ . Here the imaginary part of  $\tilde{\chi}(\omega)$  vanishes. The medium has become transparent where it would be completely opaque without the influence of the control beam. In the transparency window the real part of  $\tilde{\chi}(\omega)$  increases linearly with a steep slope, indicating that the medium is extremely dispersive. As a consequence of equation (17) the group velocity of the probe light is significantly reduced.

is violated. For a large detuning we can ignore the  $\frac{1}{4}|\Omega_c|^2$  term in the susceptibility (43). We obtain the simple Lorentzian (28) of an ordinary medium, with the combined line width  $\gamma = \gamma_1 + \gamma_2$ . Outside the narrow transparency window of EIT, the absorption of the medium has even slightly increased, because the control beam couples the medium atoms to a second dissipative transition process. The maximally tolerable detuning for transparency is proportional to the group velocity, since  $v_g$  is proportional to  $|\Omega_c|^2$  for sufficiently slow light. In practice the detuning is usually limited by  $\epsilon v_g \omega_0 / c$  with  $\epsilon$  in the order of a few  $10^{-3}$ . The transparency window concerns slow light in moving media, because of the Doppler effect. An atom with velocity  $u$  causes a Doppler detuning of  $u \omega_0 / c$ . If we fix the spectral range in the laboratory frame, the maximally tolerable velocity is  $\epsilon v_g$ . EIT is velocity-selective. Figure 3 illustrates the spectral susceptibility.

## 5 Dark-state dynamics

Suppose that a dominant and monochromatic control beam has, after relaxation, prepared the atom in the stationary state (39). How will the atom evolve when the control and probe strengths vary [14]? First we note that the state (39) is statistically pure [to quadratic order in the small quantities (34) and (38)] so that

$$\text{tr}\{\hat{\rho}^2\} = 1. \quad (45)$$

When the purity condition (45) is satisfied the density matrix contains a single state vector [8]

$$\hat{\rho} = |\psi_0\rangle\langle\psi_0| \quad (46)$$

with, in our case,

$$|\psi_0\rangle = \hat{U}_0 N_0 \left( |1\rangle - \frac{\Omega_p}{\Omega_c} |2\rangle + \frac{2(\omega - \omega_0)}{|\Omega_c|^2} \Omega_p |3\rangle \right). \quad (47)$$

The stationary state does not depend on the relaxation rates but only on the parameters of the Hamiltonian (36). Remarkably, even when the parameters vary, the state is protected from further relaxation, as long as the  $|3\rangle$  component is small,

$$\rho_{33} = \langle 3 | \hat{\rho} | 3 \rangle \ll 1. \quad (48)$$

Once the atom is in a pure state with sparsely populated top level, the purity (45) does not change during the evolution (9),

$$\text{dtr}\{\hat{\rho}^2\} = 2\text{tr}\{\hat{\rho} \text{d}\hat{\rho}\} = 4[\gamma_1(1 - \rho_{11}) + \gamma_2(1 - \rho_{22})] \rho_{33} \text{d}t. \quad (49)$$

The pure state so protected is called a dark state [15]. Although dark states are initially prepared due to the relaxation of the atomic dipoles, having so adapted to the light fields, they are no longer prone to dissipation.

Suppose that the control and the probe strengths vary. How does a dark state follow the light? In the case (48) the state of the atom is dominated by its components in the subspace spanned by the two lower levels  $|1\rangle$  and  $|2\rangle$ . If we find a state  $|\psi\rangle$  that describes correctly the dynamics (9) in this subspace, the third component  $\langle 3 | \psi \rangle$  must be correct as well, to leading order in  $\rho_{33}$ . The lower ranks enslave the top level. Since the relaxation processes (31) do not operate within the lower subspace, we can ignore dissipation entirely, to find the dominant state of the atom. We represent both control and probe light in terms of variable Rabi frequencies,

$$\Omega_c e^{-i\omega_c t} = \frac{\kappa_{23}}{\hbar} E_c^{(+)}, \quad \Omega_p e^{-i\omega_0 t} = \frac{\kappa_{13}}{\hbar} E_p^{(+)}, \quad (50)$$

defined here with respect to the atomic transition frequencies  $\omega_c = \omega_{23}$  and  $\omega_0 = \omega_{13}$ . We write down the state vector

$$|\psi\rangle = \hat{U}_0 N \left( |1\rangle - \frac{\Omega_p}{\Omega_c} |2\rangle + \frac{2N_0^2}{\Omega_c^*} i\partial_t \frac{\Omega_p}{\Omega_c} |3\rangle \right) \quad (51)$$

with the abbreviations

$$\hat{U}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-i\omega_{12}t} & 0 \\ 0 & 0 & e^{-i\omega_0t} \end{bmatrix}, \quad (52)$$

$$\frac{\Omega_p}{\Omega_c} = \left| \frac{\Omega_p}{\Omega_c} \right| e^{i\theta}, \quad (53)$$

$$N_0 = \left( 1 + \frac{|\Omega_p|^2}{|\Omega_c|^2} \right)^{-1/2}, \quad (54)$$

$$N = N_0 \exp \left( -i \int \frac{|\Omega_p|^2 d\theta}{|\Omega_p|^2 + |\Omega_c|^2} \right). \quad (55)$$

In a stationary regime under the conditions (34) and (38) the vector (51) agrees with the dark state (47). We see from the properties

$$\partial_t N = -N N_0^2 \frac{\Omega_p^*}{\Omega_c^*} \partial_t \frac{\Omega_p}{\Omega_c}, \quad \partial_t N \frac{\Omega_p}{\Omega_c} = N N_0^2 \partial_t \frac{\Omega_p}{\Omega_c}, \quad (56)$$

that  $|\psi\rangle$  satisfies the differential equation

$$i\hbar \partial_t |\psi\rangle = \hat{H} |\psi\rangle + i\hbar \partial_t \langle 3|\psi\rangle |3\rangle. \quad (57)$$

Consequently, the vector (51) describes correctly the dynamics of the atom in the lower-level subspace. Therefore, the atom remains in the dark state (51), as long as the atom's evolution never leads to an overpopulation at the top level  $|3\rangle$ . The initial relaxation-dominated regime has prepared the dark state, but later the atom follows dynamically without relaxation [14].

We calculate the matter polarization (10) generated by the dark states of the atoms that constitute the medium. The positive-frequency component of  $P$  is

$$\begin{aligned} P^{(+)} &= \frac{n_A}{2} \kappa_{31} \langle 3|\psi\rangle \langle \psi|1\rangle \\ &= \frac{n_A}{2} \kappa_{31} e^{-i\omega_0t} N_0^4 \frac{2}{\Omega_c^*} i\partial_t \frac{\Omega_p}{\Omega_c} \\ &= n_A \frac{\kappa_{31}^2}{\hbar} \frac{N_0^4}{|\Omega_c|^2} \left( i\partial_t - \omega_0 - i \frac{(\partial_t |\Omega_c|)}{|\Omega_c|} + \dot{\theta}_c \right) E_p^{(+)}, \end{aligned} \quad (58)$$

with  $\theta_c = \arg \Omega_c$ . Assume, for simplicity, that  $\Omega_c$  is real. Otherwise we can easily incorporate the phase  $\theta_c$  of the control field in the phase of the electric field without affecting the wave equation (4), as long as  $\theta_c$  varies slowly compared with the optical frequency  $\omega_0$ . We adopt the definition (41) of the group index  $\alpha$ , and get

$$\frac{1}{\varepsilon_0} \partial_t^2 P^{(+)} \approx -N_0^4 \alpha 2\omega_0 \left( i\partial_t - \omega_0 - i \frac{\dot{\alpha}}{2\alpha} \right) E_p^{(+)}. \quad (59)$$

We approximate

$$2\omega_0(i\partial_t - \omega_0) E_p^{(+)} \approx (i\partial_t + \omega_0)(i\partial_t - \omega_0) E_p^{(+)} = -(\partial_t^2 + \omega_0^2) E_p^{(+)}, \quad (60)$$

and obtain from the general wave equation (4) an equation that is valid for both the positive and the negative frequency component of the probe light,

$$\left[ \partial_t^2 - c^2 \partial_z^2 + N_0^4 \left( \partial_t \alpha \partial_t + \alpha \omega_0^2 \right) \right] E_p = 0. \quad (61)$$

The dark-state dynamics may lead to a non-linear effect of the medium, described by the  $N_0^4$  factor in the wave equation (61). However, when the probe is significantly weaker than the control light, the medium responds linearly,

$$\left( \partial_t (1 + \alpha) \partial_t - c^2 \partial_z^2 + \alpha \omega_0^2 \right) E_p = 0. \quad (62)$$

This wave equation governs the propagation of slow light in a regime of linear response and undisturbed dark-state dynamics.

## 6 Slow-light pulses

Consider a pulse of probe light in an EIT medium with variable group index (41). Suppose that the group velocity (42) does not vary much over the scale of an optical wavelength ( $0.5 \times 10^{-6} \text{m}$ ) or an optical cycle ( $10^{-15} \text{s}$ ). In this case we could apply the Hamiltonian theory (16) of a fictitious light particle to predict the position of the pulse peak. Because particle trajectories must not split, a slowly varying group index cannot cause reflection. Suppose that the pulse is traveling to the right. Then the pulse will continue to do so, and we can express the slow-light wave as

$$E_p = \mathcal{E} \exp(ikz - i\omega t) + \mathcal{E}^* \exp(-ikz + i\omega t), \quad k = \frac{\omega}{c}, \quad (63)$$

assuming that the electric-field amplitude  $\mathcal{E}$  is slowly varying compared with the rapid optical oscillations. We approximate

$$\begin{aligned} \exp(-ikz + i\omega t) \partial_t^2 E_p^{(+)} &\approx \left( -\omega^2 - 2i\omega \partial_t \right) \mathcal{E}, \\ \exp(-ikz + i\omega t) \partial_t E_p^{(+)} &\approx -i\omega \mathcal{E}, \\ \exp(-ikz + i\omega t) \partial_z^2 E_p^{(+)} &\approx \left( -k^2 + 2ik \partial_t \right) \mathcal{E}. \end{aligned} \quad (64)$$

and get from the wave equation (62)

$$\begin{aligned} -2i\omega(1 + \alpha) \partial_t \mathcal{E} &\approx \left( (1 + \alpha)\omega^2 + i\omega \dot{\alpha} - c^2 k^2 + 2ikc^2 \partial_z - \alpha \omega_0^2 \right) \mathcal{E} \\ &= \left( 2i\omega c \partial_z + i\omega \dot{\alpha} + \alpha(\omega^2 - \omega_0^2) \right) \mathcal{E} \\ &= 2i\omega \left( c \partial_z \mathcal{E} + \frac{\dot{\alpha}}{2} \mathcal{E} \right) \end{aligned} \quad (65)$$

when the carrier frequency  $\omega$  is equal to the transparency resonance  $\omega_0$ . We thus obtain the propagation equation [14]

$$(1 + \alpha) \partial_t \mathcal{E} + c \partial_z \mathcal{E} + \frac{\dot{\alpha}}{2} \mathcal{E} = 0. \quad (66)$$

In order to understand the principal behavior of ordinary slow-light pulses, we consider two cases — a spatially varying yet time-independent group index and a spatially uniform yet time-dependent  $\alpha$ .

When the group index does not change in time, the propagation equation (66) has the simple solution

$$\mathcal{E}(t, z) = \mathcal{E}_0\left(t - \int \frac{dz}{v_g}\right), \quad (67)$$

in terms of the group velocity (42). At each space point  $z$  the pulse raises and falls in precisely the same way. However, because light is slowed down, the spatial shape of the pulse shrinks by a factor of  $v_g/c$  compared with the pulse length in vacuum, for example by  $10^{-7}$  for a group velocity of 30m/s. The intensity of the pulse is unaffected, despite the enormous pulse shortening, and the pulse energy has gone into the amplification of the control beam.

Consider the other extreme, a spatially uniform EIT medium with adjustable group velocity. In this case, the solution of the propagation equation (66) is

$$\mathcal{E}(t, z) = \mathcal{E}_0\left(z - \int v_g dt\right) \sqrt{v_g/c}. \quad (68)$$

The pulse envelope  $\mathcal{E}$  propagates again with the group velocity  $v_g$  but the pulse length is not changed. However, the spectrum of the pulse around the carrier frequency  $\omega_0$  shrinks by a factor of  $v_g/c$ . Additionally, the intensity drops by  $v_g/c$  as well. The ratio between the control (41) and the pulse intensity (68) remains large,

$$(1 + \alpha) \frac{|E_c|^2}{|\mathcal{E}_0|^2} = \left(|E_c|^2 + \frac{n_A}{2} \frac{\kappa_{13}^2}{\kappa_{12}^2} \frac{\hbar\omega_0}{\varepsilon_0}\right) |\mathcal{E}_0|^{-2}, \quad (69)$$

even in the limit of a vanishing control field when  $v_g$  vanishes, as long as  $n_A$  is large (for a sufficiently dense medium). Therefore, the reaction of the probe field is remarkably consistent with the requirements for undisturbed dark-state dynamics [14]. One can freeze light without losing control [2, 3].

## 7 Effective field theory

After having studied two examples of pulse propagation in an EIT medium, we develop an effective field theory of slow light. We generalize the wave equation (62) to three-dimensional space and calibrate the electric field strength in appropriate units,

$$E_p(t, \mathbf{x}) = \left(\frac{\hbar}{\varepsilon_0}\right)^{1/2} \omega_0 \varphi(t, \mathbf{x}). \quad (70)$$

We introduce the Lagrangian

$$\mathcal{L} = \frac{\hbar}{2} \left( (1 + \alpha)(\partial_t \varphi)^2 - c^2(\nabla \varphi)^2 - \alpha \omega_0^2 \varphi^2 \right) \quad (71)$$

and see that the wave equation (62) is the resulting Euler–Lagrange equation. We have chosen the prefactor of the Lagrangian (71) such that  $\mathcal{L}$  agrees with the free-field Lagrangian (1) for zero  $\alpha$  and frequencies around  $\omega_0$ . Therefore we regard  $\mathcal{L}$  as the effective Lagrangian of slow light.

Let us use the Lagrangian (71) to calculate the energy balance of slow light. According to Noether's theorem [16] we obtain the energy density

$$I = \frac{\hbar}{2} \left( (1 + \alpha)(\partial_t \varphi)^2 + c^2(\nabla \varphi)^2 + \alpha \omega_0^2 \varphi^2 \right) \quad (72)$$

and the energy flux (Poynting vector)

$$\mathbf{S} = -\hbar c^2 (\partial_t \varphi)(\nabla \varphi). \quad (73)$$

As a consequence of the wave equation (62) we obtain the energy balance

$$\partial_t I + \nabla \cdot \mathbf{S} = \frac{\hbar \dot{\alpha}}{2} (\dot{\varphi}^2 + \omega_0^2 \varphi^2). \quad (74)$$

Temporal changes in the control field, modifying the group index (41), do not conserve energy. In fact, the experiment [2] indicates that the control beam can amplify light stored in an EIT medium with zero group velocity. In the experiment [2], slow light enters the EIT sample and is then frozen inside by turning off the control field. Switching on the control releases the stored light. The pulse emerges with an intensity that depends on the control field and that may exceed the initial intensity, in agreement with equation (68). Clearly, this phenomenon is only possible if energy is indeed transferred from the control beam to the probe light.

## 8 Moving media

An EIT medium slows down light, because the medium is extremely dispersive, reacting differently to the different frequency components of a pulse. The peak position of the pulse is the place where the frequency components interfere constructively. By slightly modifying the phase velocity of each component the medium influences strongly their interference, slowing down the pulse dramatically.

The extreme spectral sensitivity of slow light can be also applied to observe optical phenomena in moving media, caused by the Doppler effect. A uniformly moving medium would not present an interesting case, though, because uniform motion just produces a global frequency shift. However, slow light is a superb tool in detecting non-uniform motion such as rotation [17]. To understand the principal effect of slow light in moving media, we modify the Lagrangian (71) to account for the Doppler effect. We assume that (71) is valid in frames co-moving with the medium and transform back to the laboratory frame. We note that  $(\partial_t \varphi)^2 - c^2(\nabla \varphi)^2$  is a Lorentz invariant and focus on the first-order Doppler effect in the  $\alpha(\partial_t \varphi)^2$  term, assuming the realistic case of non-relativistic medium velocities. We replace  $\partial_t$  by  $\partial_t + \mathbf{u} \cdot \nabla$  and neglect the term quadratic in  $u$ . In this way we obtain the Lagrangian of slow light in a moving medium

$$\mathcal{L} = \mathcal{L}_0 - \frac{\alpha}{c^2} \mathbf{S} \cdot \mathbf{u}, \quad \mathbf{S} = -\hbar c^2 (\partial_t \varphi)(\nabla \varphi) \quad (75)$$

in terms of the Lagrangian  $\mathcal{L}_0$  for the medium at rest (71). We see that the flow  $\mathbf{u}$  couples to the Poynting vector  $\mathbf{S}$  of slow light, similar to the Röntgen interaction of



moving dipoles in electromagnetic fields [18]. We obtain from  $\mathcal{L}$  the Euler–Lagrange equation

$$\left(\partial_t(1+\alpha)\partial_t - c^2\nabla^2 + \alpha\omega_0^2 + \partial_t\alpha\mathbf{u}\cdot\nabla + \nabla\cdot\alpha\mathbf{u}\partial_t\right)\varphi = 0 \quad (76)$$

with the differential operators acting on everything to the right. For frequencies near the EIT resonance  $\omega_0$  we represent the positive-frequency part  $\varphi^{(+)}$  of  $\varphi$  as

$$\varphi^{(+)} = \varphi_0 e^{-i\omega_0 t} \quad (77)$$

and perform similar approximations as in Section 5. We obtain the Schrödinger-type equation

$$\left[i\frac{\lambda}{v_g}\left(\partial_t - \frac{v_g}{2v_g}\right) + \frac{1}{2}\right]\varphi_0 = \frac{1}{2}\left(-i\lambda\nabla + \frac{\alpha}{c}\mathbf{u}\right)^2\varphi_0 - \frac{\alpha^2 u^2}{2c^2}\varphi_0 \quad (78)$$

with the effective Planck constant reduced by  $2\pi$

$$\lambda = \frac{c}{\omega_0}. \quad (79)$$

The flow has a two-fold effect: On one hand, the velocity  $\mathbf{u}$  appears similar to an effective vector potential, for example as the magnetic vector potential acting on an electron wave, and, on the other hand, the hydrodynamic pressure proportional to  $u^2$  acts similarly to an electric potential. A vortex flow will generate the optical equivalent of the Aharonov–Bohm effect, see reference [19] for details.

A moving slow-light medium is also equivalent to an effective gravitational field [4]. Consider monochromatic light at exact resonance frequency  $\omega_0$ . In this case, we can write the wave equation (76) in the form of a Klein–Gordon equation in a curved space-time,

$$\partial_\mu \left(f^{\mu\nu} \partial_\nu \varphi^{(+)}\right) = 0, \quad (80)$$

with  $\partial_\nu = (\partial_t/c, \nabla)$  and

$$f^{\mu\nu} = \sqrt{-g} g^{\mu\nu} = \begin{bmatrix} 1 & \alpha\mathbf{u}/c \\ \alpha\mathbf{u}/c & -\mathbf{1} \end{bmatrix}. \quad (81)$$

Here  $g^{\mu\nu}$  represents the effective space-time metric experienced by monochromatic slow light in a moving medium, to first order in  $u/c$ . We easily find the determinant  $g$  of the inverse of  $g^{\mu\nu}$  from the relation

$$\det f = -g^2/g = -g = -\left(1 + \alpha^2 \frac{u^2}{c^2}\right). \quad (82)$$

The effective space-time line element  $ds^2$  is, up to a conformal factor,

$$ds^2 = c^2 dt^2 + 2\alpha dt \mathbf{u} \cdot d\mathbf{x} - d\mathbf{x}^2, \quad (83)$$

resembling the line element of a moving coordinate system,

$$ds^2 = (c^2 - u^2)dt^2 + 2dt \mathbf{u} \cdot d\mathbf{x} - d\mathbf{x}^2, \quad (84)$$

for example of a rotating system. The parameter  $\alpha$  quantifies the degree to which the motion of the medium is transferred to the propagation of light in the medium, the degree of dragging. The group index is thus equivalent to Fresnel's dragging coefficient [17, 20]. For slow light  $\alpha$  is very large indeed. Therefore, slow light is able to sense minute flow variations. Even subtle quantum flows imprint phase shifts onto slow light that are detectable using phase-contrast microscopy [21].

Sound waves in a fluid experience the flow as an effective space-time metric as well [22]. The acoustical line element is proportional to the element (84) of a moving system of coordinates, with two crucial differences: The flow is not subject to the rigidity of moving coordinate systems and, more importantly, in the acoustic metric the speed of light,  $c$ , is replaced by the speed of sound. A supersonic flow surpasses the sound barrier and can, under suitable circumstances, generate an artificial event horizon where the flow speed  $u$  reaches  $c$ . Here it is necessary that the  $g_{00}$  element of the metric vanishes. In the slow-light metric (83) the all-important term  $-u^2 dt^2$  is missing in  $g_{00}$ , at least to the level of approximation we are considering here. We obtain a term proportional to  $-u^2 dt^2$  when we include effects of higher-order Doppler detuning. The critical velocity is then of the order of  $c/\sqrt{\alpha}$ . To observe the quantum effects of light generated by a horizon we would need to employ a steep profile of the flow speed. This causes a severe problem, because the Doppler detuning will exceed the transparency window of EIT. The Doppler effect plays a beneficial role in the sensitivity of slow light to motion, but it will also cause significant light absorption when one attempts to reach an artificial event horizon. The medium will certainly turn black, but not into a black hole. However, one could employ a spatially varying profile of the group index to create an interface that resembles an event horizon for slow light and that avoids this problem [23]. Slow light offers a variety of options for interesting experiments exploiting the analogs of light in media with effects in other areas of physics, and new ideas continue to emerge.

## 9 Summary

Light has been slowed down dramatically [1] or even stopped completely [2, 3]. To understand how this feat has been achieved, we studied the physics behind slow light, starting from basic first principles of the light-matter interaction. We first turned to ordinary optical media, so as to later contrast them with slow-light media based on Electromagnetically-Induced Transparency. We studied slow light in two regimes — in a stationary regime both dominated and limited by relaxation and the control light, and in a dynamic regime almost free from dissipation. Then we analyzed the typical behavior of slow-light pulses, before developing an effective field theory of slow light that we have subsequently generalized to moving media.

It is certainly amazing how much a clever combination of atomic physics and optics can achieve, but it is also important to understand the principal limits of the techniques applied. These limits depend on the details of the physics behind the scene. We have tried to elucidate the details of slow light without going into too much detail, using models that are simple, but not too simple.

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